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The generalized 4-connectivity of hierarchical cubic networks

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ABSTRACT

Let $S \subseteq V(G)$ and $\kappa_G(S)$ denote the maximum number k of edge-disjoint trees $T_1, T_2, \cdots, T_k$ in G such that $V(T_i)\bigcap V(T_j) = S$ for any $i, j \in \{1, 2, \cdots, k\}$ and $i \neq j$. For an integer r with $2 \leq r \leq n$, the *generalized r-connectivity* of a graph G is defined as $\kappa_r(G) = min\{\kappa_G(S)|S \subseteq V(G)$ and $|S|= r\}$. In fact, $\kappa_2(G)$ is exactly the traditional connectivity of G. In this paper, we focus on $\kappa_4(HCN_n)$ of the hierarchical cubic network HCN_n and obtain that $\kappa_4(HCN_n) = n$ for $n \geq 3$. As a corollary, we obtain that $\kappa_3(HCN_n) = n$ for $n \geq 3$.

1. Introduction

An interconnection network is usually modeled by a connected graph $G = (V, E)$, where nodes represent processors and edges represent communication links between processors. The connectivity is one of the important parameters to evaluate the reliability and fault tolerance of a network. *The connectivity* $\kappa(G)$ of a graph G is defined as the minimum number of vertices whose deletion results in a disconnected graph. Whitney [20] provides another definition of connectivity. For any subset $S = \{u, v\} \subseteq V(G)$, let $\kappa_G(S)$ denote the maximum number of internally disjoint paths between u and v in G. Then $\kappa(G) = min\{\kappa_G(S)|S \subseteq V(G)$ and $|S| = 2\}$. As a generalization of the traditional connectivity, the *generalized r-connectivity* was introduced by Hager et al. [8] in 1985.

Let $S \subseteq V(G)$ and $\kappa_G(S)$ denote the maximum number k of edge-disjoint trees $T_1, T_2, \ldots, T_k$ in G such that $V(T_i)\bigcap V(T_j) = S$ for any $i, j \in \{1, 2, \ldots, k\}$ and $i \neq j$. For an integer r with $2 \leq r \leq n$, the *generalized r-connectivity* of a graph G is defined as $\kappa_r(G) = min\{\kappa_G(S)|S \subseteq V(G)$ and $|S| = r\}$. This is a parameter that can measure the reliability of a network G to connect any r vertices in G. The generalized 2-connectivity is exactly the traditional connectivity. Li et al. [10] derived that it is NP-complete for a general graph G to decide whether there are k internally disjoint trees connecting S, where k is a fixed integer, and $S \subseteq V(G)$. There are some known results [12,14,18] regarding the bounds of generalized connectivity and the relationship between connectivity and generalized connectivity. In addition, there are some known results about generalized r-connectivity for some special classes of graphs. For example, Chartrand et al. [2] studied the generalized connectivity of complete graphs; Li et al. [13] first studied the generalized 3-connectivity of Cartesian product graphs, then Li et al. [15] also studied the generalized 3-connectivity of graph products; Li et al. [11] studied the generalized connectivity of the complete bipartite graphs, Lin et al. [19] studied the generalized 4-connectivity of hypercubes and Zhao et al. studied the generalized 4-connectivity of exchanged hypercubes [25]. Zhao et al. had gotten the generalized 3-connectivity of the regular networks with the property that each vertex has exactly two outside neighbors [26], the (n, k)-bubble-sort graphs [27], the (n, k)-star graphs and alternating group graphs [23] and the Caylay

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graph generated by complete graph and wheel graph [24]. As the Cayley graph has some attractive properties to design interconnection networks, Li et al. [17] studied the generalized 3-connectivity of star graphs and bubble-sort graphs and Li et al. [16] studied the generalized 3-connectivity of the Cayley graph generated by trees and cycles. For more results about the recursive graph and Cayley graph, one can refer to [3] and [9], respectively. So far, there are few results about $\kappa_r(G)$ for $r = 4$ and almost all known results are about $r = 3$. In this paper, we obtain that $\kappa_4(HCN_n) = n$ for $n \geq 3$. As a corollary, we obtain that $\kappa_3(HCN_n) = n$ for $n \geq 3$.

The paper is organized as follows. In Section 2, some terminologies and notations are introduced. In Section 3, the generalized 4-connectivity of the hierarchical cubic network is determined. As a corollary, the generalized 3-connectivity of the hierarchical cubic network can be obtained directly. In Section 4, the paper is concluded.

2. Preliminary

2.1. Terminologies and notations

Let $G = (V, E)$ be a simple and undirected graph. Let $|V(G)|$ denote the order of the graph G. Let $V' \subseteq V(G)$, then $G[V']$ is the subgraph of G whose vertex set is V' and whose edge set consists of all edges of G which have both ends in V'. For a vertex $v \in V(G)$, the set of neighbors of v in a graph G is denoted by $N_G(v)$ and $N_G[v] = N_G(v) \cup \{v\}$. Let $d_G(v)$ denote the number of edges incident with v and $\delta(G)$ denote the *minimum degree* of the graph G. A graph is said to be *k-regular* if for any vertex v of G, $d_G(v) = k$. Two xy- paths P and Q in G are *internally disjoint* if they have no common internal vertices, that is, $V(P) \bigcap V(Q) = \{x, y\}$. Let $Y \subseteq V(G)$ and $X \subset V(G) \setminus Y$, the (X, Y)-paths is a family of internally disjoint paths starting at a vertex $x \in X$, ending at a vertex $y \in Y$ and whose internal vertices belong neither to X nor to Y. If $X = \{x\}$, then the (X, Y)-paths is a family of internal disjoint paths whose starting vertex is x and the terminal vertices are distinct in Y, which is referred to as a *k-fan* from x to Y. For terminologies and notations not defined here, refer to [1].

Let $[n] = \{1, 2, \ldots, n\}$. Let V_n be the set of binary sequence of length n, i.e., $V_n = \{x_1x_2 \cdots x_n | x_i \in \{0, 1\}$ and $1 \leq i \leq n\}$. For $x = x_1x_2 \cdots x_n \in V_n$, let $x^l = x_1 \cdots x_{l-1}\bar{x}_l x_{l+1} \cdots x_n$ and $\bar{x} = \bar{x}_1\bar{x}_2 \cdots \bar{x}_n \in V_n$, which is called the complement of x, where $\bar{x}_i \in \{0, 1\} \backslash \{x_i\}$ for each $i \in [n]$.

The hypercube is one of the most fundamental interconnection networks. An n-dimensional hypercube $Q_n = (V, E)$ is an undirected graph with $|V| = 2^n$ and $|E| = n2^{n-1}$. Each vertex can be represented by an n-bit binary string. There is an edge between two vertices whenever their binary string representation differs in only one bit position. The Hamming distance, denoted by $d_H(u, v)$, between any two vertices u and v of Q_n is the number of different positions between the binary strings of u and v. It is easy to see that two vertices u and v of the hypercube Q_n are adjacent if and only if $d_H(u, v) = 1$. The hierarchical cubic network was introduced by Ghose and Desai in [7], which can feasibly be implemented with thousands or more processors, while retaining some good properties of the hypercubes, such as regularity, symmetry and logarithmic diameter. Next, we will introduce the definition of the hierarchical cubic network.

2.2. The n-dimensional hierarchical cubic network HCN_n

The *n-dimensional hierarchical cubic network HCN_n* can be decomposed into 2^n clusters, say $C_1, C_2, \ldots, C_{2^n}$, and each cluster is isomorphic to an n-dimensional hypercube Q_n. Any node $u \in V(HCN_n)$ is identified by a unique $2n$-bit binary string, denoted by $u = (c(u), p(u))$, as an id. Each id contains two parts: n-bit cluster-id $c(u)$ and n-bit node-id $p(u)$. An edge in a cluster is called *a cube edge*, say $E_{cu}(HCN_n)$, and an edge connecting two nodes in two distinct clusters is called a *cross edge*, denoted by $E_{cr}(HCN_n)$. The set of edges that connects two distinct clusters C_i and C_j is denoted by $E_{cr}(C_i, C_j)$, where $i, j \in [2^n]$. For $u, v \in V(HCN_n)$, let $u = (c(u), p(u))$ and $v = (c(v), p(v))$. There exists an edge $uv \in E(HCN_n)$ if and only if uv belongs to one of the following conditions:

(1) $E_{cu}(HCN_n) = \{uv | c(u) = c(v)$ and $d_H(p(u), p(v)) = 1\}$,

(2) $E_{cr}(HCN_n) = \{uv |$ if $c(u) = p(u)$, then $c(v) = p(v) = \overline{c(u)}$, otherwise, $c(u) = p(v)$ and $p(u) = c(v)\}$.

By the definition of hierarchical cubic network, HCN_n is an $(n + 1)$-regular network. For any vertex v of HCN_n, it has exactly one neighbor outside the cluster which v belongs to, which is called the *outside neighbor* of v and denoted by v'. An 2-dimensional hierarchical cubic network HCN_2 is shown as Fig. 1, where the red edges represent the cross edges of HCN_2.

There are some known results about HCN_n, for which one can refer to [4–7,21,22,28] etc. for the detail. By the definition of the hierarchical cubic network HCN_n, the following result can be obtained.

Lemma 1. *Let $C_1, C_2, \ldots, C_{2^n}$ be the 2^n clusters of HCN_n for $n \geq 3$, then the following results hold.*

(1) *For $i \in [2^n]$, let $v \in V(C_i)$ with $c(v) = p(v)$. The outside neighbors of distinct vertices in $V(C_i) \setminus \{v\}$ belong to different clusters of HCN_n. In addition, if $u \in V(C_i) \setminus \{v\}$ with $p(u) = \overline{p(v)}$, the outside neighbor of u belongs to the same cluster as that of v.*

(2) *For $u \in V(C_i)$ and $v \in V(C_j)$, there are two cross edges between C_i and C_j for $i \neq j$ and $i, j \in [2^n]$ if and only if $c(u) = \overline{c(v)}$; otherwise there is only one cross edge.*

(3) *No two vertices in the same cluster of HCN_n have a common outside neighbor.*

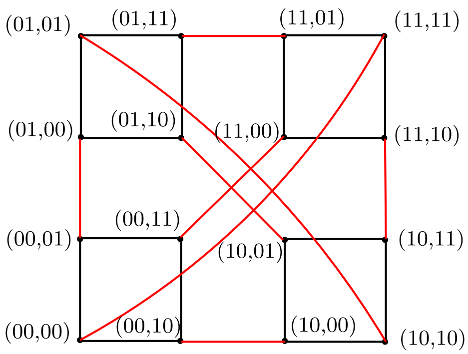

Fig. 1. The 2-dimensional hierarchical cubic network HCN_2.

Proof. (1) Let $v_1, v_2 \in V(C_i) \setminus \{v\}$ and $v_1 \neq v_2$. By the definition of HCN_n, $c(v_1) \neq p(v_1)$, $c(v_2) \neq p(v_2)$ and $p(v_1) \neq p(v_2)$. Thus, the outside neighbors of v_1 and v_2 are $(p(v_1), c(v_1))$ and $(p(v_2), c(v_2))$, respectively. Since $p(v_1) \neq p(v_2)$, the outside neighbors of v_1 and v_2 belong to different clusters of HCN_n. Since $v \in V(C_i)$ and $c(v) = p(v)$, the outside neighbor of v is $(\overline{c(v)}, \overline{p(v)})$. Let $u \in V(C_i) \setminus \{v\}$, then $c(u) \neq p(u)$ and the outside neighbor of u is $(p(u), c(u))$. When $p(u) = \overline{p(v)}$, the outside neighbors of u and v belong to the same cluster of HCN_n.

(2) By (1), the result can be obtained directly.

(3) Let $u, v \in V(C_i)$ for $i \in [2^n]$ and assume that they have a common outside neighbor, say w, then u and v are the two outside neighbors of w, which is a contradiction. □

Lemma 2. *Let $C_1, C_2, \ldots, C_{2^n}$ be the 2^n clusters of HCN_n for $n \geq 3$, then for any vertex $v \in V(C_i)$ and $i \in [2^n]$, $|N_{C_i}[v]| = n+1$ and the outside neighbors of vertices in $N_{C_i}[v]$ belong to different clusters of HCN_n.*

Proof. Without loss of generality, let $v \in V(C_1)$. C_1 is isomorphic to the n-dimensional hypercube Q_n, which is n-regular, thus $|N_{C_1}[v]| = n+1$. As $n \geq 3$, for any $v_1, v_2 \in N_{C_1}[v]$, $p(v_1) \neq \overline{p(v_2)}$. By (1) of Lemma 1, the outside neighbors of vertices in $N_{C_1}[v]$ belong to different clusters of HCN_n. □

Lemma 3. *Let $C_1, C_2, \ldots, C_{2^n}$ be the 2^n clusters of HCN_n and $H = HCN_n[\bigcup_{j=1}^{k} V(C_{i_j})]$ for $i_j \in [2^n]$, $k \geq 1$ and $n \geq 3$, then H is connected.*

Proof. Without loss of generality, let $H = HCN_n[\bigcup_{j=1}^{k} V(C_j)]$. By (2) of Lemma 1, there is at least one cross edge between any two distinct clusters of HCN_n. Thus, H is connected. □

3. The generalized 4-connectivity of the hierarchical cubic network HCN_n

In this section, we will study the generalized 4-connectivity of hierarchical cubic networks. To prove the main result, the following results are useful.

Lemma 4 ([1]). *Let G be a k-connected graph, and let x and y be a pair of distinct vertices in G. Then there exist k internally disjoint paths $P_1, P_2, \ldots, P_k$ in G connecting x and y.*

Lemma 5 ([1]). *Let $G = (V, E)$ be a k-connected graph, and let X and Y be subsets of $V(G)$ of cardinality at least k. Then there exists a family of k pairwise disjoint (X, Y)-paths in G.*

Lemma 6 ([1]). *Let $G = (V, E)$ be a k-connected graph, let x be a vertex of G, and let $Y \subseteq V(G) \setminus \{x\}$ be a set of at least k vertices of G. Then there exists a k-fan in G from x to Y. That is, there exists a family of k internally disjoint (x, Y)-paths whose terminal vertices are distinct in Y.*

The following result is about the connectivity of the hypercube Q_n.

Lemma 7 ([1]). $\kappa(Q_n) = n$ *for* $n \geq 2$.

The following result is about the generalized 4-connectivity of the hypercube Q_n.

Theorem 1 ([19]). $\kappa_4(Q_n) = n - 1$ *for* $n \geq 2$.

The following result is about the upper bound of $\kappa_k(G)$ for a connected graph G.

Lemma 8 ([19]). *Let G be a connected graph of order n with minimum degree δ. Then $\kappa_k(G) \le \delta$ for $2 \le k \le n$. In particular, if there are two adjacent vertices of degree δ, then $\kappa_k(G) \le \delta - 1$ for $3 \le k \le n$. Moreover, the upper bounds are sharp in both cases.*

The following result is about the relationship between $\kappa_k(G)$ and $\kappa_{k-1}(G)$ of a regular graph G.

Lemma 9 ([19]). *Let G be an r-regular graph. If $\kappa_k(G) = r-1$, then $\kappa_{k-1}(G) = r-1$, where $k \ge 4$.*

To prove the generalized 4-connectivity of the n-dimensional hierarchical cubic network HCN_n for $n \ge 3$, the following lemmas are useful.

Lemma 10. *Let $C_1, C_2, \ldots, C_{2^n}$ be the 2^n clusters of HCN_n for $n \ge 3$. Let $S = \{x, y, z, w\} \subseteq V(HCN_n)$ such that $|S \bigcap V(C_i)| = 3$ and $|S \bigcap V(C_j)| = 1$ for distinct $i, j \in [2^n]$, then there are n internally disjoint trees connecting S in HCN_n.*

Proof. Without loss of generality, let $|S \bigcap V(C_1)| = 3$ and $|S \bigcap V(C_2)| = 1$. Let $\{x, y, z\} \subseteq V(C_1)$ and $w \in V(C_2)$. See Fig. 2. Recall that $v = (c(v), p(v))$ for each $v \in V(HCN_n)$. As $x \ne z$, assume that $p^n(x) \ne p^n(z)$ and let $p(x) = a_1a_2\cdots a_{n-1}0$ and $p(z) = b_1b_2\cdots b_{n-1}1$. As C_i is a copy of Q_n for each $i \in [2^n]$, we assume that the nth digit of $p(y)$ is 0. Divide C_1 along the nth digit of the node-id into two copies of Q_{n-1}, denoted by Q^0_{n-1} and Q^1_{n-1}, respectively. Thus, $x, y \in V(Q^0_{n-1})$ and $z \in V(Q^1_{n-1})$. By Lemma 7, $\kappa(Q^0_{n-1}) = n-1$, then there are $n-1$ internally disjoint paths $P_1, P_2, \ldots, P_{n-1}$ between x and y in Q^0_{n-1}. Let $x_i \in V(P_i)$ such that $y_i \in V(Q^1_{n-1}) \setminus \{z\}$, where y_i is the neighbor of x_i in Q^1_{n-1} and $1 \le i \le n-1$. This can be done as P_is are internally disjoint for $1 \le i \le n-1$. Let $X = \{x_1, x_2, \ldots, x_{n-1}\}$ and $Y = \{y_1, y_2, \ldots, y_{n-1}\}$. By Lemma 7, $\kappa(Q^1_{n-1}) = n-1$. By Lemma 6, there are $n-1$ internally disjoint paths $P'_1, P'_2, \ldots, P'_{n-1}$ from z to Y in Q^1_{n-1}. Let $\widehat{T_i} = P_i \bigcup x_iy_i \bigcup P'_i$ for each $i \in [n-1]$, then $n-1$ internally disjoint trees $\widehat{T_i}$s that connecting x, y and z are obtained in C_1.

Note that $X = \{x_1, x_2, \ldots, x_{n-1}\}$, it is possible that $x \in X$ or $y \in X$. To avoid duplication, we just consider the case that $x \notin X$ and $y \notin X$. Let $X' = X \cup \{x, y\}$. By (1) of Lemma 1, the outside neighbors of vertices in X' belong to different clusters of HCN_n. Thus, there is at most one vertex of X' with the outside neighbor belonging to C_2. To obtain the main result, the following two cases are considered.

Case 1. There is one vertex in X' with the outside neighbor belonging to C_2.

Without loss of generality, let $x'_1 \in V(C_2)$, $x'_i \in V(C_{i+1})$ for $2 \le i \le n-1$, $x' \in V(C_{n+1})$ and $y' \in V(C_{n+2})$. By (2) of Lemma 1, there is an edge $w_iw'_i \in E_{cr}(C_{i+1}, C_2)$ such that $w_i \in V(C_{i+1})$ and $w'_i \in V(C_2)$ for $2 \le i \le n-1$. Let $W' = \{x'_1, w'_2, \ldots, w'_{n-1}\}$, then the following subcases are considered depending on the outside neighbor z' of z.

Subcase 1.1. $z' \in V(C_2)$.

As any vertex of HCN_n has exactly one outside neighbor, $z' \notin W'$. Let $W = W' \cup \{z'\} = \{x'_1, w'_2, \ldots, w'_{n-1}, z'\}$, thus $|W| = n$.

If $w \notin W$, by (2) of Lemma 1, $w' \notin \cup_{i=1}^{n} V(C_i)$. Without loss of generality, let $w' \in V(C_{n+3})$. See Fig. 2. By Lemma 7, $\kappa(C_2) = n$. By Lemma 4, there are n internally disjoint paths $W_1, W_2, \ldots, W_n$ from w to W such that $x'_1 \in W_1$, $w'_i \in W_i$ for $2 \le i \le n-1$ and $z' \in W_n$. As C_{i+1} is connected, there is a path $\widehat{P_i}$ between x'_i and w_i in C_{i+1} for $2 \le i \le n-1$. By Lemma 3, $HCN_n[\cup_{i=n+1}^{n+3} V(C_i)]$ is connected, so it contains a tree T that connects x', y' and w'. Let $T_1 = \widehat{T_1} \cup W_1 \cup x_1x'_1$, $T_i = \widehat{T_i} \cup \widehat{P_i} \cup W_i \cup x_ix'_i \cup w_iw'_i$ for $2 \le i \le n-1$ and $T_n = W_n \cup T \cup xx' \cup yy' \cup zz' \cup ww'$, then n internally disjoint S-trees T_is for $1 \le i \le n$ are obtained in HCN_n.

If $w \in W$, let $\widehat{W} = (W \setminus \{w\}) \cup \{v\}$ for $v \in V(C_2)$ and $v' \in V(C_{n+3})$. By (2) of Lemma 1, this can be done. Similar as $w \notin W$, n internally disjoint S-trees T_is for $1 \le i \le n$ can be obtained in HCN_n.

Subcase 1.2. $z' \in V(C_{i+1})$ for some $i \in [n-1] \setminus [1]$.

Without loss of generality, let $z' \in V(C_3)$. See Fig. 3. Since $z', x'_2 \in V(C_3)$, by (2) of Lemma 1, $y'_2 \notin \cup_{i=1}^{n+2} V(C_i)$. Without loss of generality, let $y'_2 \in V(C_{n+3})$. By (2) of Lemma 1, there are edges $aa' \in E_{cr}(C_{n+3}, C_2)$ and $bb' \in E_{cr}(C_{n+2}, C_2)$ such that $a \in V(C_{n+3})$, $b \in V(C_{n+2})$, and $a', b' \in V(C_2)$. Recall that there is an edge $w_iw'_i \in E_{cr}(C_{i+1}, C_2)$ such that $w_i \in V(C_{i+1})$ and $w'_i \in V(C_2)$ for $3 \le i \le n-1$. Let $W = \{x'_1, a', w'_3, \ldots, w'_{n-1}, b'\}$. By Lemma 7, $\kappa(C_2) = n$. By Lemma 4, there are n internally disjoint paths $W_1, W_2, \ldots, W_n$ from w to W such that $x'_1 \in W_1$, $a' \in W_2$, $w'_i \in W_i$ for $3 \le i \le n-1$ and $b' \in W_n$. As C_{i+1} is connected, there is a path $\widehat{P_i}$ between x'_i and w_i in C_{i+1} for $3 \le i \le n-1$ and there is a path P between y'_2 and a in C_{n+3}. By Lemma 3, $HCN_n[V(C_3 \cup C_{n+1} \cup C_{n+2})]$ is connected, thus it contains a tree T that connects x', y', z' and b. Let $T_1 = \widehat{T_1} \cup W_1 \cup x_1x'_1$, $T_2 = \widehat{T_2} \cup W_2 \cup P \cup \cup y_2y'_2 \cup aa'$, $T_i = \widehat{T_i} \cup \widehat{P_i} \cup W_i \cup x_ix'_i \cup w_iw'_i$ for $3 \le i \le n-1$ and $T_n = W_n \cup T \cup bb' \cup xx' \cup yy' \cup zz'$, then n internally disjoint S-trees are obtained in HCN_n.

Subcase 1.3. $z' \in V(HCN_n) \setminus \cup_{i=1}^{n} V(C_i)$.

Without loss of generality, let $z' \in V(C_{n+1})$. By (2) of Lemma 1, there is an edge $aa' \in E_{cr}(C_{n+2}, C_2)$ such that $a \in V(C_{n+2})$ and $a' \in V(C_2)$. See Fig. 4. Recall that there is an edge $w_iw'_i \in E_{cr}(C_{i+1}, C_2)$ such that $w_i \in V(C_{i+1})$ and $w'_i \in V(C_2)$ for $2 \le i \le n-1$. Let $W = \{x'_1, w'_2, w'_3, \ldots, w'_{n-1}, a'\}$. By Lemma 7, $\kappa(C_2) = n$. By Lemma 4, there are n internally disjoint paths $W_1, W_2, \ldots, W_n$ from w to W such that $x'_1 \in W_1$, $w'_i \in W_i$ for $2 \le i \le n-1$ and $a' \in W_n$. As C_{i+1} is connected, there is a path $\widehat{P_i}$ between x'_i and w_i for $2 \le i \le n-1$ in C_{i+1}. By Lemma 3, $HCN_n[V(C_{n+1} \cup C_{n+2})]$ is connected, thus it contains a tree T connecting x', y', z' and a. Let $T_1 = \widehat{T_1} \cup W_1 \cup x_1x'_1$, $T_i = \widehat{T_i} \cup \widehat{P_i} \cup W_i \cup x_ix'_i \cup w_iw'_i$ for $2 \le i \le n-1$ and $T_n = W_n \cup T \cup aa' \cup xx' \cup yy' \cup zz'$, then the result is obtained.

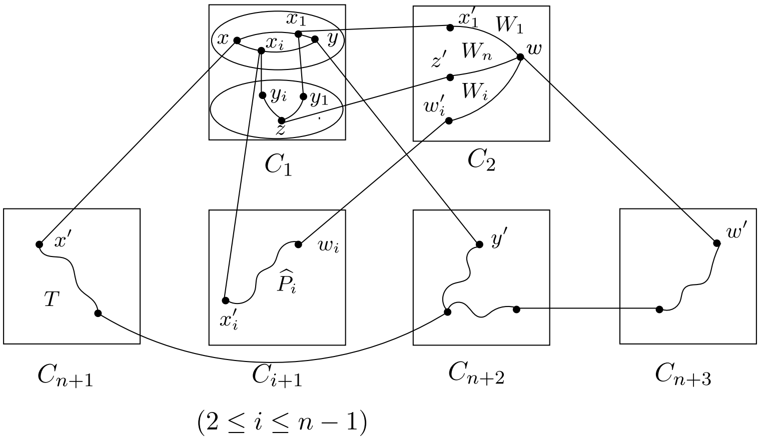

Fig. 2. The illustration of $z' \in V(C_2)$.

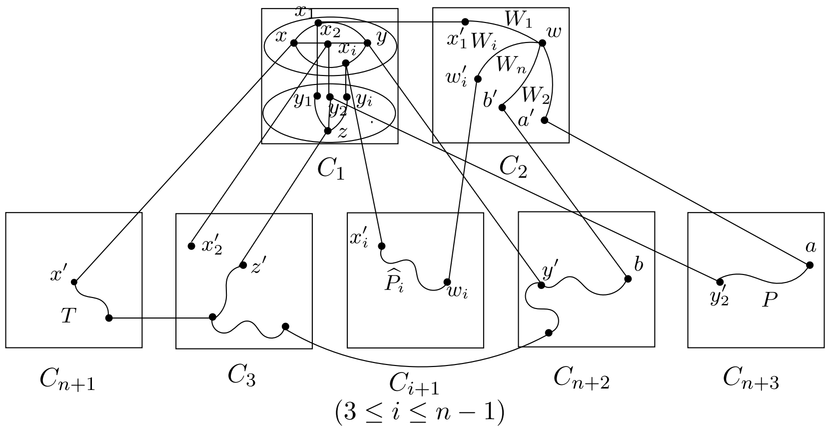

Fig. 3. The illustration of $z' \in V(C_3)$.

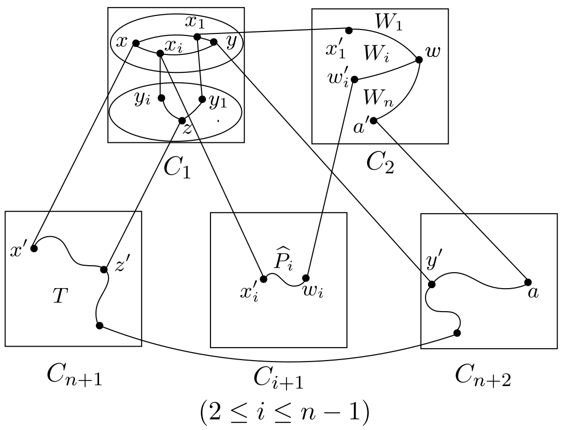

Fig. 4. The illustration of $z' \in V(C_{n+1})$.

Case 2. None of the vertices in X' have their outside neighbors belonging to C_2.

Without loss of generality, let $x_i' \in V(C_{i+2})$ for $1 \leq i \leq n-1$, $x' \in V(C_{n+2})$ and $y' \in V(C_{n+3})$. To prove the result, the following subcases are considered.

Subcase 2.1. $z' \in V(C_2)$.

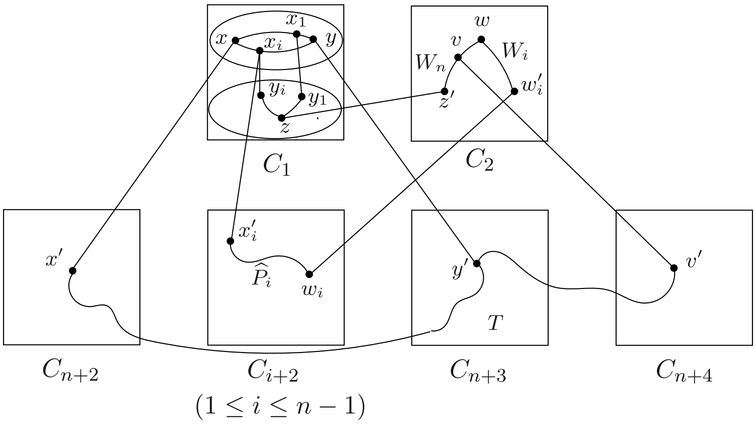

Fig. 5. The illustration of Subcase 2.1.1.

By (2) of Lemma 1, there is an edge $w_iw_i' \in E_{cr}(C_{i+2}, C_2)$ such that $w_i \in V(C_{i+2})$ and $w_i' \in V(C_2)$ for $1 \leq i \leq n-1$. Let $W = \{w_1', w_2', \ldots, w_{n-1}', z'\}$. By Lemma 7, $\kappa(C_2) = n$. By Lemma 4, there are n internally disjoint paths $W_1, W_2, \ldots, W_n$ from w to W such that $w_i' \in W_i$ for $1 \leq i \leq n-1$ and $z' \in W_n$. As C_{i+2} is connected, there is a path $\widehat{P_i}$ between x_i' and w_i in C_{i+2} for $1 \leq i \leq n-1$. Consequently, we just consider $z' \neq w$ and $w_i' \neq w$ for each $i \in [n-1]$ by the location of w' as the discussions for $z' = w$ or $w_i' = w$ for some $i \in [n-1]$ are similar.

Subcase 2.1.1. $w' \in V(C_1)$

In this case, $ww', zz' \in E_{cr}(C_1, C_2)$ and $z', w \in V(C_2)$. By (1) of Lemma 1, $p(z') = \overline{p(w)}$. Thus, $d_H(z', w) = n \geq 3$. Recall that W_n is the path from z' to w in C_2, so there is a vertex $v \in V(W_n) \setminus \{z', w\}$. See Fig. 5. As $ww', zz' \in E_{cr}(C_1, C_2)$, by (2) of Lemma 1, $v' \notin \cup_{i=1}^{n+1} V(C_i)$. That is, $v' \in \cup_{i=n+2}^{2^n} V(C_i)$. As $HCN_n[\cup_{i=n+2}^{2^n} V(C_i)]$ is connected, it contains a tree T connecting x', y' and v'. Let $T_i = \widehat{T_i} \cup \widehat{P_i} \cup W_i \cup x_ix_i' \cup w_iw_i'$ for $1 \leq i \leq n-1$ and $T_n = T \cup W_n \cup xx' \cup yy' \cup zz' \cup vv'$, then n internally disjoint S-trees T_is for $1 \leq i \leq n$ are obtained in HCN_n.

Subcase 2.1.2. $w' \in V(C_{i+2})$ for some $i \in [n-1]$

Without loss of generality, let $w' \in V(C_3)$. See Fig. 6. By (2) of Lemma 1, there is an edge $aa' \in E_{cr}(C_{n+2}, C_3)$ such that $a \in V(C_{n+2})$ and $a' \in V(C_3)$. Let $S = \{x_1', w'\}$ and $T = \{w_1, a'\}$. By Lemma 5, there are two internally disjoint (S, T)-paths, say $\widehat{P}$ and P, such that $\widehat{P}$ is the path from x_1' to w_1 and P is the path from w' to a'. Let $H = HCN_n[V(C_{n+2} \cup C_{n+3})]$. By Lemma 3, H is connected. Thus, there is a tree T connecting x', a and y' in H. Let $T_1 = \widehat{T_1} \cup \widehat{P} \cup W_1 \cup x_1x_1' \cup w_1w_1'$, $T_i = \widehat{T_i} \cup \widehat{P_i} \cup W_i \cup x_ix_i' \cup w_iw_i'$ for $2 \leq i \leq n-1$, and $T_n = W_n \cup P \cup T \cup \cup xx' \cup yy' \cup zz' \cup ww' \cup aa'$, then n internally disjoint S-trees T_is for $1 \leq i \leq n$ are obtained in HCN_n.

Subcase 2.1.3. $w' \in \cup_{i=n+2}^{2^n} V(C_i)$

By Lemma 3, $HCN_n[\cup_{i=n+2}^{2^n} V(C_i)]$ is connected and it has a tree T connecting x', y' and w'. Let $T_i = \widehat{T_i} \cup \widehat{P_i} \cup W_i \cup x_ix_i' \cup w_iw_i'$ for $1 \leq i \leq n-1$ and $T_n = W_n \cup T \cup xx' \cup yy' \cup zz' \cup ww'$, then n internally disjoint S-trees are obtained in HCN_n.

Subcase 2.2. $z' \in V(C_{i+2})$ for some $i \in [n-1]$.

Without loss of generality, let $z' \in V(C_3)$, then $x_1x_1', zz' \in E_{cr}(C_1, C_3)$. See Fig. 7. By (2) of Lemma 1, $y_1' \notin \cup_{i=1}^{n+3} V(C_i)$. Without loss of generality, let $y_1' \in V(C_{n+4})$. By (2) of Lemma 1, there are edges $aa' \in E_{cr}(C_{n+3}, C_2)$ and $bb' \in E_{cr}(C_{n+4}, C_2)$ such that $a \in V(C_{n+3}), b \in V(C_{n+4})$ and $a', b' \in V(C_2)$. Recall that there is an edge $w_iw_i' \in E_{cr}(C_{i+2}, C_2)$ such that $w_i \in V(C_{i+2})$ and $w_i' \in C_2$ for $2 \leq i \leq n-1$. Let $W = \{b', w_2', w_3', \ldots, w_{n-1}', a'\}$. By Lemma 7, $\kappa(C_2) = n$. By Lemma 4, there are n internally disjoint paths $W_1, W_2, \ldots, W_n$ from w to W such that $b' \in W_1$, $w_i' \in W_i$ for $2 \leq i \leq n-1$ and $a' \in W_n$. As C_i is connected for each $i \in [2^n]$, there is a path P between y_1' and b in C_{n+4} and there is a path $\widehat{P_i}$ between x_i' and w_i in C_{i+2} for $2 \leq i \leq n-1$. Let $H = HCN_n[V(C_3 \cup C_{n+2} \cup C_{n+3})]$. By Lemma 3, H is connected. Thus, there is a tree T connecting x', y', z' and a in H. Let $T_1 = \widehat{T_1} \cup P \cup W_1 \cup y_1y_1' \cup bb'$, $T_i = \widehat{T_i} \cup \widehat{P_i} \cup W_i \cup x_ix_i' \cup w_iw_i'$ for $2 \leq i \leq n-1$ and $T_n = W_n \cup T \cup aa' \cup xx' \cup yy' \cup zz'$, then n internally disjoint S-trees are obtained in HCN_n.

Subcase 2.3. $z' \in \cup_{i=n+2}^{2^n} V(C_i)$.

Let $H = HCN_n[\cup_{i=n+2}^{2^n} V(C_i)]$. By Lemma 3, H is connected. By (2) of Lemma 1, there are edges $w_iw_i' \in E_{cr}(C_{i+2}, C_2)$ such that $w_i \in V(C_{i+2})$ and $w_i' \in V(C_2)$ for $1 \leq i \leq n-1$ and $aa' \in E_{cr}(C_{n+3}, C_2)$ such that $a \in V(C_{n+3})$ and $a' \in V(C_2)$. Let $W = \{w_1', w_2', w_3', \ldots, w_{n-1}', a'\}$. By Lemma 7, $\kappa(C_2) = n$. By Lemma 4, there are n internally disjoint paths $W_1, W_2, \ldots, W_n$ from w to W such that $w_i' \in W_i$ for $1 \leq i \leq n-1$ and $a' \in W_n$. As C_{i+2} is connected, it contains a path $\widehat{P_i}$ between x_i' and w_i in C_{i+2} for $1 \leq i \leq n-1$. As H is connected, it contains a tree T connecting x', y', z' and a. Let $T_i = \widehat{T_i} \cup \widehat{P_i} \cup W_i \cup x_ix_i' \cup w_iw_i'$ for $1 \leq i \leq n-1$ and $T_n = W_n \cup T \cup aa' \cup xx' \cup yy' \cup zz'$, thus n internally disjoint S-trees are obtained in HCN_n. □

Lemma 11. *Let $C_1, C_2, \ldots, C_{2^n}$ be the 2^n clusters of HCN_n for $n \geq 3$. Let $S = \{x, y, z, w\} \subseteq V(HCN_n)$ such that $|S \bigcap V(C_i)| = 2$ and $|S \bigcap V(C_j)| = 2$ for distinct $i, j \in [2^n]$, then there are n internally disjoint trees connecting S in HCN_n.*

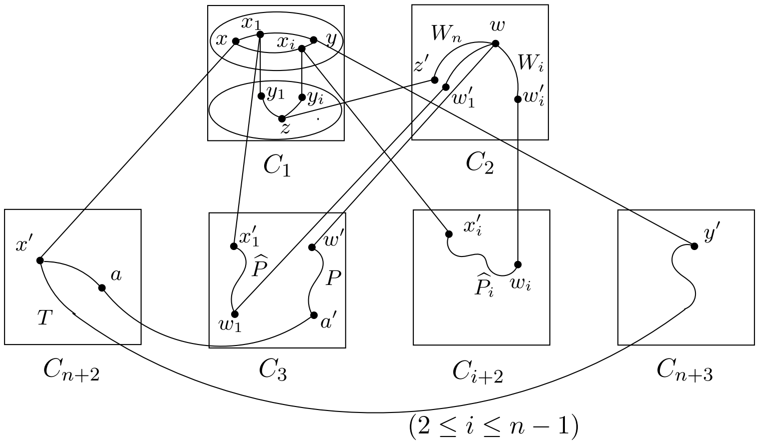

Fig. 6. The illustration of Subcase 2.1.2.

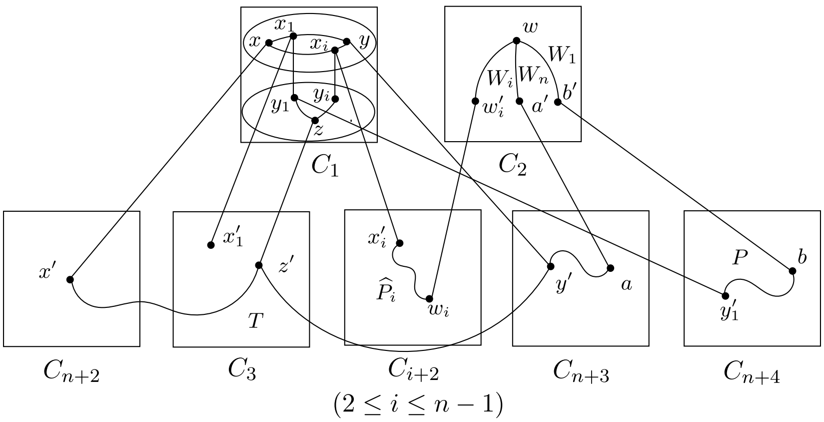

Fig. 7. The illustration of Subcase 2.2.

Proof. Without loss of generality, let $|S\bigcap V(C_1)| = 2$ and $|S\bigcap V(C_2)| = 2$. Let $\{x, y\} \subseteq V(C_1)$ and $\{z, w\} \subseteq V(C_2)$. See Fig. 8. By Lemma 7, $\kappa(C_1) = \kappa(C_2) = n$, then there are n internally disjoint paths $P_1, P_2, \ldots, P_n$ between x and y in C_1 and n internally disjoint paths $P'_1, P'_2, \ldots, P'_n$ between z and w in C_2. Let $x_i \in V(P_i)\cap N(x)$ and $z_i \in V(P'_i)\cap N(z)$ for $1 \le i \le n$. Let $\widehat{X} = \{x, x_1, x_2, \ldots, x_n\}$ and $\widehat{Z} = \{z, z_1, z_2, \ldots, z_n\}$. Choose n vertices from $\widehat{X}$, denoted by X, such that the outside neighbor of any vertex in X does not belong to C_2. Similarly, choose n vertices from $\widehat{Z}$, denoted by Z, such that the outside neighbor of any vertex in Z does not belong to C_1. By Lemma 2, this can be done. Without loss of generality, let $X = \{x_1, x_2, \ldots, x_n\}$ and $Z = \{z_1, z_2, \ldots, z_n\}$. Let $X' = \{x'_1, x'_2, \ldots, x'_n\}$ and $Z' = \{z'_1, z'_2, \ldots, z'_n\}$, where x'_i and z'_i are the outside neighbors of x_i and z_i, respectively. By Lemma 2, the vertices in $X'(resp. Z')$ belong to different clusters of HCN_n. Without loss of generality, let $x'_i \in V(C_{i+2})$ for $1 \le i \le n$. By the location of the vertices in Z', the following two cases need to be considered.

Case 1. The vertices in $X' \cup Z'$ belong to different clusters of HCN_n.

Without loss of generality, let $z'_i \in V(C_{n+2+i})$ for $i \in [n]$. As $2^n \ge 2n + 2$ for $n \ge 3$, this can be done. By Lemma 3, $HCN_n[V(C_i \bigcup C_{n+2+i})]$ is connected for each $i \in [n]$. Then there is a path $\widehat{P_i}$ between x'_i and z'_i in $HCN_n[V(C_i \bigcup C_{n+2+i})]$ for each $i \in [n]$. Let $T_i = P_i \bigcup P'_i \cup \widehat{P_i} \cup x_i x'_i \bigcup z_i z'_i$ for each $i \in [n]$, thus n internally disjoint S-trees T_is for $1 \le i \le n$ are obtained in HCN_n.

Case 2. There exists an element of X' which belongs to the same cluster with some element of Z'.

Without loss of generality, let x'_i and z'_i belong to the same cluster for $1 \le i \le m$, where $1 \le m \le n$. In addition, let $z'_i \in V(C_{n+2-m+i})$ for $m + 1 \le i \le n$. As C_i is connected, there is a path $\widehat{\widehat{P_i}}$ between x'_i and z'_i in C_i for $1 \le i \le m$. In addition, there is a path $\widehat{P_i}$ between x'_i and z'_i in $HCN_n[V(C_i \bigcup C_{n+2-m+i})]$, as it is connected for $m + 1 \le i \le n$. Let $T_i = P_i \bigcup P'_i \cup \widehat{P_i} \cup x_i x'_i \bigcup z_i z'_i$ for each $i \in [n]$, then n internally disjoint S-trees T_is for $1 \le i \le n$ are obtained in HCN_n. □

Recall that the n-dimensional hierarchical cubic network HCN_n can be decomposed into 2^n clusters, say $C_1, C_2, \ldots, C_{2^n}$. As C_i is isomorphic to an n-dimensional hypercube Q_n for each $i \in [2^n]$, by Lemma 7, $\kappa(C_i) = n$. Let $x, y \in V(C_1)$, then

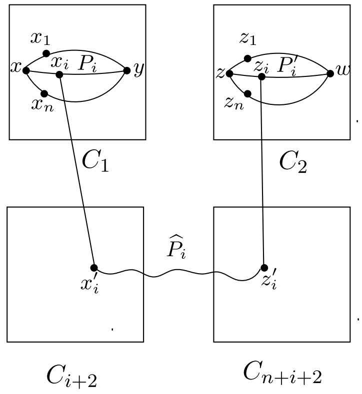

Fig. 8. The illustration of Case 1.

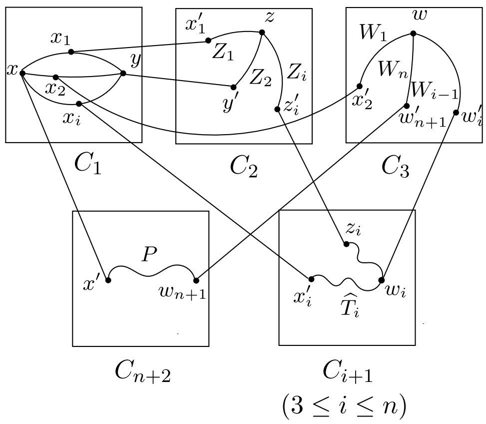

Fig. 9. The illustration of $y' \in V(C_2)$.

there are n internally disjoint paths $P_1, P_2, \ldots, P_n$ between x and y in C_1. By the known result, we have the following lemma.

Lemma 12. *Let $C_1, C_2, \ldots, C_{2^n}$ be the 2^n clusters of HCN_n for $n \ge 3$. Let $S = \{x, y, z, w\} \subseteq V(HCN_n)$ such that $x, y \in V(C_1), z \in V(C_2)$ and $w \in V(C_3)$. Let $P_1, P_2, \ldots, P_n$ be the n internally disjoint paths between x and y in C_1. Let $x_i \in N(x) \cap V(P_i)$ for $i \in [n]$ and $N[x] = \{x, x_1, x_2, \ldots, x_n\}$. If there are two cross edges between $N[x]$ and $V(C_2 \cup C_3)$, then there are n internally disjoint trees connecting S in HCN_n.*

Proof. Let x', y' and x_i' be the outside neighbors of x, y and x_i for $1 \le i \le n$, respectively. By Lemma 2, the outside neighbors of vertices in $N[x]$ belong to different clusters of HCN_n. Consequently, we just consider the case for $y \notin N[x]$ as the discussion for $y \in N[x]$ is similar.

Case 1. $x_i' \in V(C_2)$ and $x_j' \in V(C_3)$ for some two distinct $i, j \in [n]$.

Without loss of generality, let $x_1' \in V(C_2), x_2' \in V(C_3), x_i' \in V(C_{i+1})$ for $3 \le i \le n$ and $x' \in V(C_{n+2})$.

If $y' \in V(C_2)$, by (2) of Lemma 1, there is an edge $z_iz_i' \in E_{cr}(C_{i+1}, C_2)$ such that $z_i \in V(C_{i+1})$ and $z_i' \in V(C_2)$ for $3 \le i \le n$. See Fig. 9. In addition, there is an edge $w_iw_i' \in E_{cr}(C_{i+1}, C_3)$ such that $w_i \in V(C_{i+1})$ and $w_i' \in V(C_3)$ for $3 \le i \le n+1$. Let $Z = \{x_1', y', z_3', \ldots, z_n'\}$ and $W = \{x_2', w_3', w_4', \ldots, w_{n+1}'\}$. By Lemma 7, $\kappa(C_2) = \kappa(C_3) = n$. By Lemma 4, there are n internally disjoint paths $Z_1, Z_2, \ldots, Z_n$ from z to Z and n internally disjoint paths $W_1, W_2, \ldots, W_n$ from w to W such that $x_1' \in Z_1, y' \in Z_2, z_i' \in Z_i$ for $3 \le i \le n, x_2' \in W_1$ and $w_i' \in W_{i-1}$ for $3 \le i \le n+1$. As C_{n+2} is connected, there is a path P between x' and w_{n+1} in C_{n+2}. In addition, there is a tree $\widehat{T}_i$ connecting x_i', z_i and w_i in C_{i+1} for $3 \le i \le n$. Let $T_1 = P_1 \cup Z_1 \cup P \cup W_n \cup x_1x_1' \cup xx' \cup w_{n+1}w_{n+1}'$, $T_2 = P_2 \cup Z_2 \cup W_1 \cup yy' \cup x_2x_2'$ and $T_i = P_i \cup \widehat{T}_i \cup Z_i \cup W_{i-1} \cup x_ix_i' \cup z_iz_i' \cup w_iw_i'$ for $3 \le i \le n$, then n internally disjoint trees connecting S are obtained in HCN_n.

If $y' \in V(C_3)$, similar as $y' \in V(C_2)$, n internally disjoint trees connecting S can be obtained in HCN_n.

If $y' \in V(C_{i+1})$ for some $3 \le i \le n$, without loss of generality, let $y' \in V(C_4)$. See Fig. 10. As $2^n > n+3$ for $n \ge 3$, there is a cluster, say C_{n+3}. By (2) of Lemma 1, there is an edge $aa' \in E_{cr}(C_{n+2}, C_2)$ such that $a \in V(C_{n+2})$ and $a' \in V(C_2)$. In addition, there are edges $vv' \in E_{cr}(C_4, C_3)$, $bb' \in E_{cr}(C_{n+3}, C_2)$, $cc' \in E_{cr}(C_{n+3}, C_3)$ and $uu' \in E_{cr}(C_4, C_{n+3})$

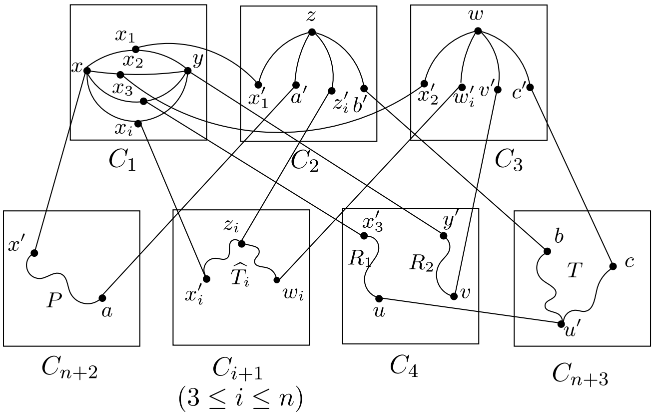

Fig. 10. The illustration of $y' \in V(C_4)$.

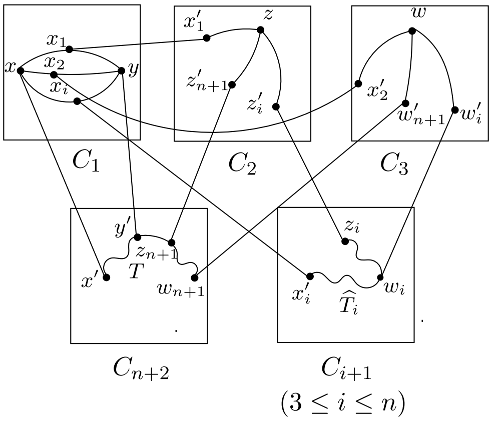

Fig. 11. The illustration of $y' \in V(C_{n+2})$.

such that $v \in V(C_4)$, $v' \in V(C_3)$, $b, c, u' \in V(C_{n+3})$, $b' \in V(C_2)$, $c' \in V(C_3)$ and $u \in V(C_4)$. In addition, there are edges $z_iz_i' \in E_{cr}(C_{i+1}, C_2)$ and $w_iw_i' \in E_{cr}(C_{i+1}, C_3)$ such that $z_i, w_i \in V(C_{i+1})$, $z_i' \in V(C_2)$ and $w_i' \in V(C_3)$ for $4 \le i \le n$. As C_i is connected for each $i \in [2^n]$, there is a path P between x' and a in C_{n+2}, there is a tree T connecting u', b and c in C_{n+3} and there is a tree T_i' connecting x_i', z_i and w_i in C_i for $4 \le i \le n$. As $\kappa(C_4) = n \ge 3$, let $S = \{x_3', y'\}$ and $T = \{u, v\}$. By Lemma 5, there are two disjoint paths from S to T, say R_1 and R_2, such that R_1 is a path from x_3' to u and R_2 is a path from y' to v. Let $Z = \{x_1', a', b', z_4', \dots, z_n'\}$ and $W = \{v', x_2', c', w_4', \dots, w_n'\}$. By Lemma 7, $\kappa(C_2) = \kappa(C_3) = n$. By Lemma 4, there are n internally disjoint paths $Z_1, Z_2, \dots, Z_n$ from z to Z and n internally disjoint paths $W_1, W_2, \dots, W_n$ from w to W such that $x_1' \in Z_1, a' \in Z_2, b' \in Z_3, z_i' \in Z_i$, $v' \in W_1, x_2' \in W_2, c' \in W_3$ and $w_i' \in W_i$ for $4 \le i \le n$. Let $T_1 = P_1 \cup x_1x_1' \cup Z_1 \cup yy' \cup R_2 \cup vv' \cup W_1$, $T_2 = P_2 \cup x_2x_2' \cup W_2 \cup xx' \cup P \cup aa' \cup Z_2$, $T_3 = P_3 \cup x_3x_3' \cup R_1 \cup uu' \cup T \cup bb' \cup cc' \cup Z_3 \cup W_3$ and let $T_i = P_i \cup \widehat{T}_i \cup x_ix_i' \cup z_iz_i' \cup w_iw_i' \cup Z_i \cup W_i$ for $4 \le i \le n$, then n internally disjoint trees connecting S are obtained in HCN_n.

If $y' \in V(C_{n+2})$. See Fig. 11. By (2) of Lemma 1, there are edges $z_iz_i' \in E_{cr}(C_{i+1}, C_2)$ and $w_iw_i' \in E_{cr}(C_{i+1}, C_3)$ such that $z_i, w_i \in V(C_{i+1})$, $z_i' \in V(C_2)$ and $w_i' \in V(C_3)$ for $3 \le i \le n+1$. Let $Z = \{x_1', z_3', z_4', \dots, z_{n+1}'\}$ and $W = \{x_2', w_3', w_4', \dots, w_{n+1}'\}$. By Lemma 7, $\kappa(C_2) = \kappa(C_3) = n$. By Lemma 4, there are n internally disjoint paths $Z_1, Z_2, \dots, Z_n$ from z to Z and n internally disjoint paths $W_1, W_2, \dots, W_n$ from w to W such that $x_1' \in Z_1, z_i' \in Z_{i-1}, x_2' \in W_1$ and $w_i' \in W_{i-1}$ for $3 \le i \le n+1$. As C_i is connected for each $i \in [2^n]$, there is a tree T connecting x', y', z_{n+1} and w_{n+1} in C_{n+2}. In addition, there is a tree $\widehat{T}_{i-1}$ connecting x_i', z_i and w_i in C_{i+1} for $3 \le i \le n$. Let $T_1 = P_1 \cup (P_2 \setminus \{x\}) \cup Z_1 \cup W_1 \cup x_1x_1' \cup x_2x_2'$, $T_{i-1} = P_i \cup \widehat{T}_{i-1} \cup Z_{i-1} \cup W_{i-1} \cup x_ix_i' \cup z_iz_i' \cup w_iw_i'$ for $3 \le i \le n$ and $T_n = T \cup Z_n \cup W_n \cup xx' \cup yy' \cup z_{n+1}z_{n+1}' \cup w_{n+1}w_{n+1}'$, then n internally disjoint trees connecting S are obtained in HCN_n.

If $y' \in V(HCN_n) \setminus \cup_{i=1}^{n+2} V(C_i)$. Without loss of generality, let $y' \in V(C_{n+3})$. See Fig. 12. By (2) of Lemma 1, there are edges $z_iz_i' \in E_{cr}(C_{i+1}, C_2)$ and $w_iw_i' \in E_{cr}(C_{i+1}, C_3)$ such that $z_i, w_i \in V(C_{i+1})$, $z_i' \in V(C_2)$ and $w_i' \in V(C_3)$ for $3 \le i \le n+1$. Let $Z = \{x_1', z_3', z_4', \dots, z_{n+1}'\}$ and $W = \{x_2', w_3', w_4', \dots, w_{n+1}'\}$. By Lemma 7, $\kappa(C_2) = \kappa(C_3) = n$. By Lemma 4, there are n internally disjoint paths $Z_1, Z_2, \dots, Z_n$ from z to Z and n internally disjoint paths $W_1, W_2, \dots, W_n$ from w to W such that $x' \in Z_1, z_i' \in Z_{i-1}, x_2' \in W_1$ and $w_i' \in W_{i-1}$ for $3 \le i \le n+1$. By Lemma 3, $HCN_n[V(C_{n+2} \cup C_{n+3})]$ is connected. Thus, it

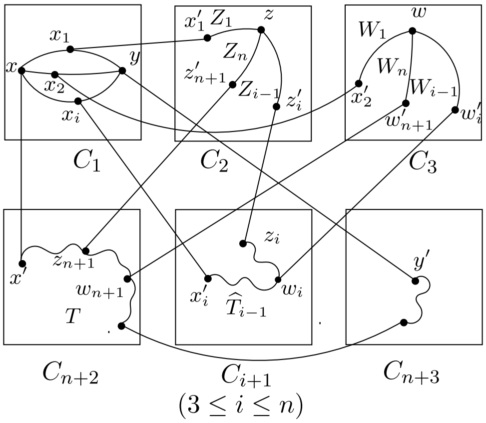

Fig. 12. The illustration of $y' \in V(C_{n+3})$.

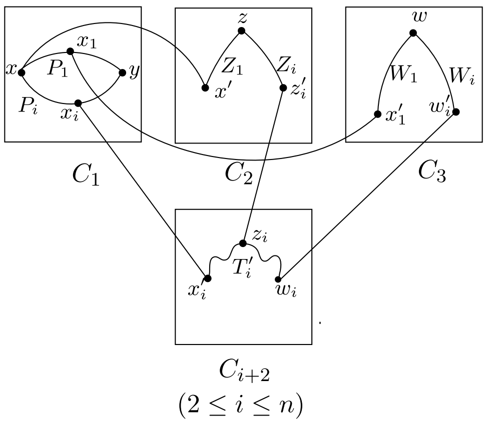

Fig. 13. The illustration of Case 2 of Lemma 12.

contains a tree T that connects x', z_{n+1}, w_{n+1} and y'. As C_{i+1} is connected for $3 \leq i \leq n+1$, there is a tree $\widehat{T}_{i-1}$ connecting x'_i, z_i and w_i in C_{i+1} for $3 \leq i \leq n+1$. Let $T_1 = P_1 \cup Z_1 \cup W_1 \cup (P_2 \setminus \{x\}) \cup x_1x'_1 \cup x_2x'_2$, $T_{i-1} = P_i \cup \widehat{T}_{i-1} \cup Z_{i-1} \cup W_{i-1} \cup x_ix'_i \cup z_iz'_i \cup w_iw'_i$ for $3 \leq i \leq n$ and $T_n = T \cup Z_n \cup W_n \cup xx' \cup z_{n+1}z'_{n+1} \cup yy' \cup w_{n+1}w'_{n+1}$, then n internally disjoint trees connecting S are obtained in HCN_n.

Case 2. $x' \in V(C_2)$ and $x'_i \in V(C_3)$ for some $i \in [n]$.

Without loss of generality, let $x' \in V(C_2)$, $x'_1 \in V(C_3)$ and $x'_i \in V(C_{i+2})$ for $2 \leq i \leq n$. See Fig. 13. By (2) of Lemma 1, there are edges $z_iz'_i \in E_{cr}(C_{i+2}, C_2)$ and $w_iw'_i \in E_{cr}(C_{i+2}, C_3)$ such that $z'_i \in V(C_2)$, $w'_i \in V(C_3)$ and $z_i, w_i \in V(C_{i+2})$ for $2 \leq i \leq n$. Let $Z = \{x', z'_2, z'_3, \ldots, z'_n\}$ and $W = \{x'_1, w'_2, w'_3, \ldots, w'_n\}$. By Lemma 7, $\kappa(C_2) = \kappa(C_3) = n$. By Lemma 4, there are n internally disjoint paths $Z_1, Z_2, \ldots, Z_n$ from z to Z and n internally disjoint paths $W_1, W_2, \ldots, W_n$ from w to W such that $x' \in Z_1$, $z'_i \in Z_i$, $x'_1 \in W_1$ and $w'_i \in W_i$ for $2 \leq i \leq n$. As C_{i+2} is connected for each $i \in [2^n]$, there is a tree T'_i connecting x'_i, z_i and w_i in C_{i+2} for $2 \leq i \leq n$. Let $T_1 = P_1 \cup Z_1 \cup W_1 \cup xx' \cup x_1x'_1$ and $T_i = P_i \cup T'_i \cup Z_i \cup W_i \cup x_ix'_i \cup z_iz'_i \cup w_iw'_i$ for $2 \leq i \leq n$, then n internally disjoint trees connecting S are obtained in HCN_n. □

Lemma 13. *Let $C_1, C_2, \ldots, C_{2^n}$ be the 2^n clusters of HCN_n for $n \geq 3$. Let $S = \{x, y, z, w\} \subseteq V(HCN_n)$ such that $x, y \in V(C_1)$, $z \in V(C_2)$ and $w \in V(C_3)$. Let $P_1, P_2, \ldots, P_n$ be the n internally disjoint paths between x and y in C_1. Let $x_i \in N(x) \cap V(P_i)$ for $i \in [n]$ and $N[x] = \{x, x_1, x_2, \ldots, x_n\}$. If there is at most one cross edge between $N[x]$ and $V(C_2 \cup C_3)$, then there are n internally disjoint trees connecting S in HCN_n.*

Proof. Let x', y' and x'_i be the outside neighbors of x, y and x_i for $1 \leq i \leq n$, respectively. By Lemma 2, the outside neighbors of vertices in $N[x]$ belong to different clusters of HCN_n. To prove the result, the following cases are considered.

Case 1. There is exactly one cross edge between $N[x]$ and $V(C_2 \cup C_3)$.

Without loss of generality, let $x'_1 \in V(C_2)$, $x'_i \in V(C_{i+2})$, and $x' \in V(C_{n+3})$ for $2 \leq i \leq n$. See Fig. 14. By (2) of Lemma 1, there are edges $z_iz'_i \in E_{cr}(C_{i+2}, C_2)$ for $2 \leq i \leq n$ and $w_iw'_i \in E_{cr}(C_{i+2}, C_3)$ for $2 \leq i \leq n+1$ such that $z_i, w_i \in V(C_{i+2})$, $z'_i \in$

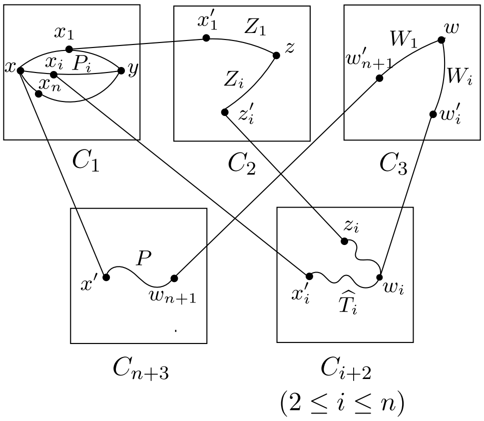

Fig. 14. The illustration of Case 1 of Lemma 13.

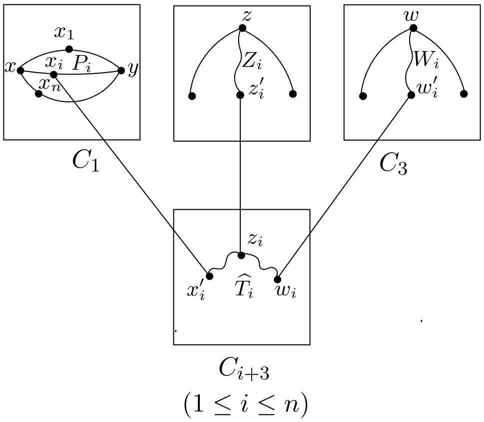

Fig. 15. The illustration of Case 2 of Lemma 13.

$V(C_2)$ and $w'_i \in V(C_3)$. As C_i is connected for each $i \in [2^n]$, there is a path P between x' and w_{n+1} in C_{n+3} and a tree $\widehat{T}_i$ connecting x'_i, z_i and w_i in C_{i+2} for $2 \le i \le n$. Let $Z = \{x'_1, z'_2, z'_3, \dots, z'_n\}$ and $W = \{w'_2, w'_3, w'_4, \dots, w'_{n+1}\}$. By Lemma 7, $\kappa(C_2) = \kappa(C_3) = n$. By Lemma 4, there are n internally disjoint paths $Z_1, Z_2, \dots, Z_n$ from z to Z and n internally disjoint paths $W_1, W_2, \dots, W_n$ from w to W such that $x'_1 \in Z_1, z'_i \in Z_i, w'_{n+1} \in W_1$, and $w'_i \in W_i$ for $2 \le i \le n$. Let $T_1 = P_1 \cup Z_1 \cup P \cup W_1 \cup x_1x'_1 \cup xx' \cup w_{n+1}w'_{n+1}$ and $T_i = P_i \cup \widehat{T}_i \cup Z_i \cup W_i \cup x_ix'_i \cup z_iz'_i \cup w_iw'_i$ for $2 \le i \le n$. Then n internally disjoint trees connecting S are obtained in HCN_n.

Case 2. There is no cross edge between $N[x]$ and $V(C_2 \cup C_3)$.

Without loss of generality, let $x'_i \in V(C_{i+3})$ for $1 \le i \le n$. See Fig. 15. By (2) of Lemma 1, there are edges $z_iz'_i \in E_{cr}(C_{i+3}, C_2)$ and $w_iw'_i \in E_{cr}(C_{i+3}, C_3)$ such that $z_i, w_i \in V(C_{i+3}), z'_i \in V(C_2)$ and $w'_i \in V(C_3)$ for $1 \le i \le n$. Let $Z = \{z'_1, z'_2, \dots, z'_n\}$ and $W = \{w'_1, w'_2, \dots, w'_n\}$. By Lemma 7, $\kappa(C_2) = \kappa(C_3) = n$. By Lemma 4, there are n internally disjoint paths $Z_1, Z_2, \dots, Z_n$ from z to Z and n internally disjoint paths $W_1, W_2, \dots, W_n$ from w to W such that $z'_i \in Z_i$ and $w'_i \in W_i$ for $1 \le i \le n$. As C_{i+3} is connected, there is a tree $\widehat{T}_i$ connecting x'_i, z_i and w_i in C_{i+3} for $1 \le i \le n$. Let $T_i = P_i \cup \widehat{T}_i \cup Z_i \cup W_i \cup x_ix'_i \cup z_iz'_i \cup w_iw'_i$ for $1 \le i \le n$, then the result is obtained. □

Lemma 14. *Let $C_1, C_2, \dots, C_{2^n}$ be the 2^n clusters of HCN_n for $n \ge 3$. Let $S = \{x, y, z, w\} \subseteq V(HCN_n)$ such that $|S \cap V(C_i)| = 1$, $|S \cap V(C_j)| = 1$, $|S \cap V(C_k)| = 1$ and $|S \cap V(C_\ell)| = 1$, i, j, k, ℓ are mutually distinct and $i, j, k, \ell \in [2^n]$, then there are n internally disjoint trees connecting S in HCN_n.*

Proof. Without loss of generality, let $|S \bigcap V(C_1)| = 1, |S \bigcap V(C_2)| = 1, |S \bigcap V(C_3)| = 1$, and $|S \bigcap V(C_4)| = 1$. Let $x \in V(C_1), y \in V(C_2), z \in V(C_3)$, and $w \in V(C_4)$, see Fig. 16. By (2) of Lemma 1 and $2^n \ge n + 4$ for $n \ge 3$, one can choose n vertices from C_1, say $x_1, x_2, \dots, x_n$, such that $x'_i \in V(C_{i+4})$, where x'_i is the outside neighbor of x_i in HCN_n and $1 \le i \le n$. Then choose n vertices $y_1, y_2, \dots, y_n$ from C_2, n vertices $z_1, z_2, \dots, z_n$ from C_3 and n vertices $w_1, w_2, \dots, w_n$ from C_4 such that $y'_i, z'_i, w'_i \in V(C_{i+4})$, where y'_i, z'_i and w'_i are the outside neighbors of y_i, z_i and w_i, respectively. Let

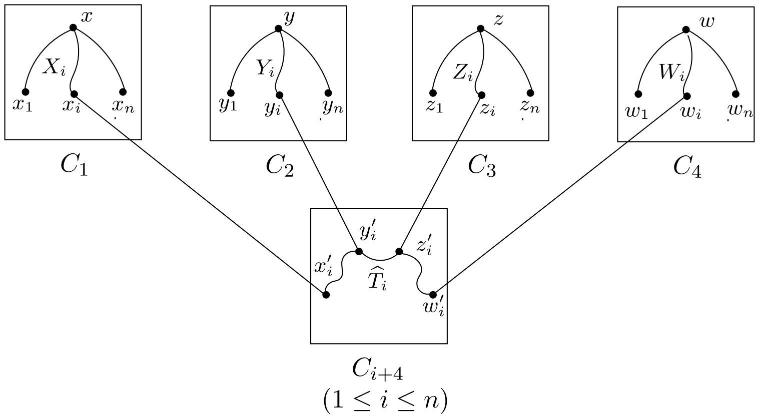

Fig. 16. The illustration of the proof of Lemma 14.

$X = \{x_1, x_2, \ldots, x_n\}, Y = \{y_1, y_2, \ldots, y_n\}, Z = \{z_1, z_2, \ldots, z_n\}$ and $W = \{w_1, w_2, \ldots, w_n\}$. By Lemma 4, there are n internally disjoint paths $X_1, X_2, \ldots, X_n$ from x to X such that $x_i \in X_i$, n internally disjoint paths $Y_1, Y_2, \ldots, Y_n$ from y to Y such that $y_i \in Y_i$, n internally disjoint paths $Z_1, Z_2, \ldots, Z_n$ from z to Z such that $z_i \in Z_i$ and n internally disjoint paths $W_1, W_2, \ldots, W_n$ from w to W such that $w_i \in W_i$, respectively. It is possible that one of the paths X_is($resp.Y_is, Z_is, W_is$) is a single vertex. As C_{i+4} is connected, there is a tree $\widehat{T_i}$ connecting x'_i, y'_i, z'_i and w'_i in C_{i+4} for each $i \in [n]$. Let $T_i = X_i \cup Y_i \cup Z_i \cup W_i \cup \widehat{T_i} \cup x_ix'_i \cup y_iy'_i \cup z_iz'_i \cup w_iw'_i$ for each $i \in [n]$. Then n internally disjoint S-trees T_is for $1 \leq i \leq n$ are obtained in HCN_n. □

Theorem 2. *Let HCN_n be an n-dimensional hierarchical cubic network, then $\kappa_4(HCN_n) = n$.*

Proof. As HCN_n is $(n+1)$-regular, by Lemma 8, $\kappa_4(HCN_n) \leq \delta - 1 = n$. To prove the result, we just need to show that $\kappa_4(HCN_n) \geq n$. Let $S = \{x, y, z, w\}$, where x, y, z and w are any four distinct vertices of HCN_n. By the symmetry of HCN_n, we prove the result by considering the following cases.

Case 1. x, y, z and w belong the same cluster of HCN_n.

Without loss of generality, let $S \subseteq V(C_1)$. Recall that C_1 is a copy of Q_n. By Theorem 1, $\kappa_4(Q_n) = n - 1$. Then there are $n-1$ internally disjoint S-trees $T_1, T_2, \ldots, T_{n-1}$ in C_1. Let x', y', z' and w' be the outside neighbors of x, y, z and w in HCN_n, respectively. Then $\{x', y', z', w'\} \subseteq V(HCN_n \setminus C_1)$. By Lemma 3, $HCN_n \setminus C_1$ is connected. Thus, there is a tree $\widehat{T_n}$ connecting x', y', z' and w' in $HCN_n \setminus C_1$. Let $T_n = \widehat{T_n} \bigcup xx' \bigcup yy' \bigcup zz' \bigcup ww'$, then $T_1, T_2, \ldots, T_n$ are n-internally disjoint S-trees in HCN_n and the result is as desired.

Case 2. x, y, z and w belong to two distinct clusters of HCN_n.

By Lemmas 10 and 11, n-internally disjoint S-trees $T_1, T_2, \ldots, T_n$ can be obtained in HCN_n.

Case 3. x, y, z and w belong to three distinct clusters of HCN_n.

Without loss of generality, let $x, y \in V(C_1), z \in V(C_2)$ and $w \in V(C_3)$. By Lemma 7, $\kappa(C_1) = n$, thus there are n internally disjoint paths $P_1, P_2, \ldots, P_n$ between x and y in C_1. Let $x_i \in N(x) \cap V(P_i)$ for $i \in [n]$ and $N[x] = \{x, x_1, x_2, \ldots, x_n\}$. By Lemma 2, the outside neighbors of vertices in $N[x]$ belong to different clusters of HCN_n. Thus, there are at most two cross edges between $N[x]$ and $V(C_2 \cup C_3)$. By Lemmas 12 and 13, n-internally disjoint S-trees $T_1, T_2, \ldots, T_n$ can be obtained in HCN_n.

Case 4. x, y, z and w belong to four distinct clusters of HCN_n.

By Lemma 14, n-internally disjoint S-trees $T_1, T_2, \ldots, T_n$ can be obtained in HCN_n.

Thus, $\kappa_4(HCN_n) = n$ and the result is desired. □

Corollary 1. *Let HCN_n be an n-dimensional hierarchical cubic network for $n \geq 3$, then $\kappa_3(HCN_n) = n$.*

Proof. By Theorem 2, $\kappa_4(HCN_n) = n$. As HCN_n is $(n+1)$-regular, by Lemma 9, $\kappa_3(HCN_n) = n$. Thus, the result holds. □

4. Concluding remarks

The hierarchical cubic network HCN_n has some attractive properties to design interconnection networks. In this paper, we focus on $\kappa_4(HCN_n)$ of the hierarchical cubic network HCN_n and obtain that $\kappa_4(HCN_n) = n$ for $n \geq 3$. As a corollary, we obtain that $\kappa_3(HCN_n) = n$ for $n \geq 3$. In future work, the generalized r-connectivity of the hierarchical cubic network for $r \geq 5$ would be an interesting problem.

Acknowledgments

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